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Quantum Stochastic Integrals as Operators

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Abstract We construct quantum stochastic integrals for the integrator being a martingale in a von Neumann algebra, and the integrand—a suitable process with values in the same algebra, as densely defined operators affiliated with the algebra. In the case of a finite algebra we allow the integrator to be an L^2-martingale in which case the integrals are L^2-martingales too.

Keywords Quantum stochastic integrals · Quantum martingales · Adapted processes

Introduction

The theory of 'general' quantum stochastic integrals (i.e. not founded on Fock space) deals mainly with the setup which can be roughly described as follows: For a von Neumann algebra $\mathcal{A}$ with a filtration $\{\mathcal{A}_t : t \geqslant 0\}$ we have a corresponding process $(X(t) : t \geqslant 0)$ with values in $L^p(\mathcal{A})$, and a corresponding process $(f(t) : t \geqslant 0)$ with values in $L^q(\mathcal{A})$, $1/p + 1/q = 1/2$, $2 \leqslant p, q \leqslant +\infty$, $L^p(\mathcal{A})$ and $L^q(\mathcal{A})$ being appropriate noncommutative L^p-spaces. Then under various specific assumptions about X and f, among which the most natural is that X is a martingale, one can define stochastic integrals $\int_a^b f\,dX$ and $\int_a^b dX\,f$ as elements of $L^2(\mathcal{A})$ (cf. [1, 4, 5, 7–10]). The advantage of this is that, for any reasonable definition of the integral, it may be approximated by integral sums of the form $\sum f(t_{i-1})[X(t_i) - X(t_{i-1})]$ (for $\int_a^b f\,dX$, the other one being defined in complete analogy). Now a product of elements from $L^p(\mathcal{A})$ and $L^q(\mathcal{A})$, is an element from $L^r(\mathcal{A})$, where $1/p + 1/q = 1/r$. In particular, the approximation holds in the case $r = 2$ (in a Hilbert space). However, this approach can be exploited also in the following quite natural setting. If we assume that the algebra $\mathcal{A}$ acts in a Hilbert space $\mathcal{H}$, and if we let f and X take their values in $\mathcal{A}$, then the integral sum belongs to $\mathcal{A}$ too, and we may ask about its behavior on

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elements of $\mathcal{H}$. This again leads us to approximation of the integral evaluated at some points of $\mathcal{H}$, i.e., we come to the notion of integrals $\int_a^b f\,dX$ and $\int_a^b dX\,f$ as operators on $\mathcal{H}$. This idea has been carried out in [3] for a particular class of martingales X defined 'canonically' in quasi-free representations of the CCR and CAR.

It turns out that it is possible to apply this point of view in the following situation: As the integrator we take a monotone or norm-continuous $\mathcal{A}$-valued process. Then we show that there exists a Riemann-Stieltjes type integral $\int_a^b f\,dX$ as a densely defined operator affiliated with the algebra $\mathcal{A}$. In the case when $\mathcal{A}$ is finite, we show the existence of both the integrals $\int_a^b f\,dX$ and $\int_a^b dX\,f$ as densely defined closed operators affiliated with $\mathcal{A}$. Moreover, we can weaken the assumptions about $(X(t))$ and allow it to be a martingale in $L^2(\mathcal{A})$, in which case the integrals above will be elements from $L^2(\mathcal{A})$ too.

Finally, let us say a few words about quantum stochastic integrals in general. In the existing theories of quantum integration, especially those of Lebesgue type, the classes of 'theoretically admissible' integrands are rather narrow and lack any *concrete* examples of processes that can be integrated. On the other hand for integrals of Riemann-Stieltjes type such examples have been provided, and, for example, it was shown how one can integrate predictable processes and how integration with respect to a quantum random time can be performed (cf. [8, 9]). In this note we take a similar approach showing the possibility of integrating either a monotone or norm-continuous process with respect to a martingale. It is worth noting that in the case of a monotone process, apparently nothing can be said about the existence of Lebesgue type integrals, while in the case of a norm-continuous process its existence can be proved only under an additional assumption.

1 Preliminaries and Notation

Throughout the paper, we assume that $\mathcal{A}$ is a von Neumann algebra acting in a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$, and that ω is a normal faithful state given by a cyclic and separating unit vector Ω in $\mathcal{H}$, i.e., $\mathcal{A}$ is represented in standard form. We suppose, further, that we have an increasing family $\{\mathcal{A}_t : t \in [0,+\infty)\}$ of von Neumann subalgebras of $\mathcal{A}$ ($\mathcal{A}_s \subset \mathcal{A}_t$ for $s \leqslant t$), called a filtration, and a corresponding family $\{\mathbb{E}_t\}$ of normal conditional expectations from $\mathcal{A}$ onto $\mathcal{A}_t$ leaving ω invariant.

A *process* in $\mathcal{A}$ or $L^2(\mathcal{A})$ is a function defined on $[0,+\infty)$ with values in $\mathcal{A}$ or $L^2(\mathcal{A})$, respectively. We shall denote by f, processes in $\mathcal{A}$, and by X, processes either in $\mathcal{A}$ or $L^2(\mathcal{A})$. Following the notation of probability theory, we shall sometimes denote a process by $(X(t) : t \geqslant 0)$ (a family of 'random variables'), and the same applies to f.

The norms in $\mathcal{H}$ and $L^2(\mathcal{A})$ will be denoted $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_2$, respectively, while $\|\cdot\|$ will stand for the operator norm.

Define on (a dense subspace of) $\mathcal{H}$ operators P_t given by:

$$P_t(x\Omega) = (\mathbb{E}_t x)\Omega, \quad x \in \mathcal{A}. \tag{1}$$

It is well-known (cf. e.g. [6] Propositions 1.1 and 1.2) that $(P_t : t \geqslant 0)$ is an increasing family of orthogonal projections in $\mathcal{H}$ with ranges $\mathcal{H}_t = \overline{\mathcal{A}_t\Omega}$. Moreover, $P_t \in \mathcal{A}_t'$, where $\mathcal{A}_t'$ is the commutant of $\mathcal{A}_t$.

A process in $\mathcal{A}$ or $L^2(\mathcal{A})$ will be called *adapted* if its value at each point $t \geqslant 0$ belongs to $\mathcal{A}_t$ or $L^2(\mathcal{A}_t)$, respectively. We call processes f in $\mathcal{A}$, and X in $L^2(\mathcal{A})$ *martingales* if for any $s,t \in [0,+\infty)$, $s \leqslant t$, the equalities

$$\mathbb{E}_s f(t) = f(s), \qquad \mathbb{E}_s X(t) = X(s)$$

hold, where in the L^2-case we use the same symbol $\mathbb{E}_s$ to denote the extension of the conditional expectation to $L^2(\mathcal{A})$. It follows that a martingale is an adapted process.

Let $(X(t) : t \in [0, +\infty))$ be a process, and let $0 \leqslant t_0 \leqslant t_1 \leqslant \ldots \leqslant t_m < +\infty$ be a sequence of points. To simplify the notation we put

$$\Delta X(t_k) = X(t_k) - X(t_{k-1}), \quad k = 1, \ldots, m.$$

Let $(X(t) : t \in [0, +\infty))$, $(f(t) : t \in [0, +\infty))$ be arbitrary processes, and let $[a, b]$ be a subinterval of $[0, +\infty)$. For a partition $\theta = \{a = t_0 < t_1 < \ldots < t_m = b\}$ of $[a, b]$ we form left and right integral sums

$$S_\theta^l = \sum_{k=1}^m \Delta X(t_k) f(t_{k-1}), \qquad S_\theta^r = \sum_{k=1}^m f(t_{k-1}) \Delta X(t_k).$$

If there exist limits (in any sense) of the above sums as θ is refined, we call them respectively the *left* and *right stochastic integrals* of f with respect to $(X(t))$, and denote

$$\lim_\theta S_\theta^l = \int_a^b dX\, f, \qquad \lim_\theta S_\theta^r = \int_a^b f\, dX.$$

This notion of integral is a weaker one than defining the integrals as the limits

$$\int_a^b dX\, f = \lim_{\|\theta\| \to 0} S_\theta^l, \qquad \int_a^b f\, dX = \lim_{\|\theta\| \to 0} S_\theta^r,$$

where $\|\theta\|$ stands for the mesh of the partition θ. A definition of this kind is standard in the classical theory of Riemann-Stieltjes as well as the theory of stochastic integrals. It is worth noticing that in noncommutative integration theory, whenever this Riemann-Stieltjes type integral is considered, its definition refers to the weaker form of the limit with the refining net of partitions (cf. [2, 8, 9]). However, under additional assumptions we shall be able to obtain the integral also in the stronger sense thus making it similar to the classical stochastic integral.

2 Integrals as Operators on $\mathcal{H}$—the Non-tracial Case

Our construction of the integral is given by the following

Theorem 1 *Let $(X(t) : t \geqslant 0)$ be a martingale in $\mathcal{A}$, and let $f : [0, \infty) \to \mathcal{A}$ be a hermitian adapted process such that f is monotone. Then for each $t > 0$ there exists a Riemann–Stieltjes type integral $\int_0^t f\, dX$, which is a densely defined operator on $\mathcal{H}$ affiliated with $\mathcal{A}$.*

Proof Fix $t > 0$, and let $\theta = \{0 = t_0 < \ldots < t_m = t\}$ be a partition of $[0, t]$. Put

$$S_\theta^r = \sum_{k=1}^m f(t_{k-1})[X(t_k) - X(t_{k-1})].$$

We want to define $\int_0^t f\, dX$ as an operator on $\mathcal{A}'\Omega$ (where $\mathcal{A}'$ is the commutant of $\mathcal{A}$) by

$$\int_0^t f dX(x'\Omega) = \lim_\theta S_\theta^r(x'\Omega), \quad x' \in \mathcal{A}',$$

as θ is refined. For this it is sufficient to show the existence of the limit $\lim_\theta S_\theta^r \Omega$, since

$$S_\theta^r x' = x' S_\theta^r.$$

We have

$$\begin{aligned}
S_\theta^r \Omega &= \sum_{k=1}^m f(t_{k-1})[X(t_k) - X(t_{k-1})]\Omega \\
&= \sum_{k=1}^m f(t_{k-1})[\mathbb{E}_{t_k} X(t) - \mathbb{E}_{t_{k-1}} X(t)]\Omega \\
&= \sum_{k=1}^m f(t_{k-1})(P_{t_k} - P_{t_{k-1}}) X(t)\Omega,
\end{aligned}$$

where the P_t are projections defined by (1). Consider the operator sum

$$\sigma_\theta^r = \sum_{k=1}^m f(t_{k-1})(P_{t_k} - P_{t_{k-1}}) = \sum_{k=1}^m (P_{t_k} - P_{t_{k-1}}) f(t_{k-1}) = (\sigma_\theta^r)^*. \tag{2}$$

To fix attention, assume that f is increasing (i.e., $f(s) \leqslant f(t)$ for $s \leqslant t$), and let $\theta' = \theta \cup \{t'\}$ for some $t_j < t' < t_{j+1}$ be a one-point refinement of θ. Then

$$\begin{aligned}
\sigma_{\theta'}^r - \sigma_\theta^r &= f(t_j)(P_{t'} - P_{t_j}) + f(t')(P_{t_{j+1}} - P_{t'}) - f(t_j)(P_{t_{j+1}} - P_{t_j}) \\
&= [f(t') - f(t_j)](P_{t_{j+1}} - P_{t'}) \\
&= (P_{t_{j+1}} - P_{t'})[f(t') - f(t_j)](P_{t_{j+1}} - P_{t'}) \geqslant 0,
\end{aligned}$$

because $P_{t_{j+1}} - P_{t'}$ is a projection commuting with $f(t_j)$ and $f(t')$, and $f(t') - f(t_j) \geqslant 0$. It follows that the net $\{\sigma_\theta^r\}$ is increasing. Furthermore, for each $k = 1, \ldots, m$ we have

$$(P_{t_k} - P_{t_{k-1}}) f(t_{k-1})^2 (P_{t_k} - P_{t_{k-1}}) \leqslant \|f(t_{k-1})\|^2 (P_{t_k} - P_{t_{k-1}}).$$

Hence

$$\begin{aligned}
\sum_{k=1}^m (P_{t_k} - P_{t_{k-1}}) f(t_{k-1})^2 (P_{t_k} - P_{t_{k-1}}) &\leqslant \sum_{k=1}^m \|f(t_{k-1})\|^2 (P_{t_k} - P_{t_{k-1}}) \\
&\leqslant c^2 \sum_{k=1}^m (P_{t_k} - P_{t_{k-1}}) = c^2 (P_t - P_0),
\end{aligned}$$

where $c = \sup_{0\leqslant s\leqslant t} \|f(s)\| = \max\{\|f(0)\|, \|f(t)\|\}$. Consequently,

$$\begin{aligned}
\|\sigma_\theta^r\|^2 = \left\|(\sigma_\theta^r)^2\right\| &= \left\| \sum_{i,j=1}^m (P_{t_j} - P_{t_{j-1}}) f(t_{j-1}) f(t_{i-1}) (P_{t_i} - P_{t_{i-1}}) \right\| \\
&= \left\| \sum_{i=1}^m (P_{t_i} - P_{t_{i-1}}) f(t_{i-1})^2 (P_{t_i} - P_{t_{i-1}}) \right\| \leqslant c^2 \|P_t - P_0\| = c^2,
\end{aligned}$$

which means that $\{\sigma_\theta^r\}$ is norm-bounded. This, together with the fact that the net is increasing, yields the existence of $\lim_\theta \sigma_\theta^r$ in the strong operator topology, in particular, there exists

$$\lim_\theta \sigma_\theta^r(X(t)\Omega) = \lim_\theta S_\theta^r \Omega.$$

It is clear that for each $x', y' \in \mathcal{A}'$ we have

$$\begin{aligned}\left(\int_0^t f\,dX\right) y'(x'\Omega) &= \lim_\theta S_\theta^r y'(x'\Omega) \\ &= y' \lim_\theta S_\theta^r(x'\Omega) = y'\left(\int_0^t f\,dX\right)(x'\Omega),\end{aligned}$$

thus $\int_0^t f\,dX$ is affiliated with $\mathcal{A}'' = \mathcal{A}$. □

It turns out that even a stronger form of integral can be obtained for norm-continuous processes.

Theorem 2 *Let $(X(t) : t \geqslant 0)$ be a martingale in $\mathcal{A}$, and let $f : [0, \infty) \to \mathcal{A}$ be a norm-continuous adapted process. Then for each $t > 0$ there exists a Riemann-Stieltjes type integral $\int_0^t f\,dX$ which is a densely defined operator on $\mathcal{H}$ affiliated with $\mathcal{A}$. Moreover, this integral is given as the limit*

$$\lim_{\|\theta\|\to 0} \sum_{k=1}^m f(t_{k-1})\Delta X(t_k)$$

on $\mathcal{A}'\Omega$.

Proof Fix $t > 0$. We shall show that the net $\{S_\theta^r \Omega\}$ is Cauchy as the mesh $\|\theta\|$ of the partition θ tends to 0. Take an arbitrary $\varepsilon > 0$, and let $\delta > 0$ be such that for each $t', t'' \in [0, t]$ with $|t' - t''| < \delta$ we have

$$\|f(t') - f(t'')\| < \frac{\varepsilon}{2\|X(t)\Omega\|_{\mathcal{H}}}.$$

Let $\theta' = \{0 = t_0 < t_1 < \ldots < t_m = t\}$ be an arbitrary partition of $[0, t]$ with $\|\theta'\| < \delta$, and let θ'' be a partition of $[0, t]$ finer than θ'. Denote by $t_0^{(k)}, t_1^{(k)}, \ldots, t_{l_k}^{(k)}$ the points of θ'' lying between t_{k-1} and t_k, such that $t_{k-1} = t_0^{(k)} < t_1^{(k)} < \ldots < t_{l_k}^{(k)} = t_k$. We then have

$$\begin{aligned}S_{\theta''}^r &= \sum_{k=1}^m \sum_{i=1}^{l_k} f(t_{i-1}^{(k)})\Delta X(t_i^{(k)}) \\ S_{\theta'}^r &= \sum_{k=1}^m f(t_{k-1})\Delta X(t_k) = \sum_{k=1}^m \sum_{i=1}^{l_k} f(t_{k-1})\Delta X(t_i^{(k)}),\end{aligned}$$

so that

$$S_{\theta''}^r - S_{\theta'}^r = \sum_{k=1}^m \sum_{i=1}^{l_k} [f(t_{i-1}^{(k)}) - f(t_{k-1})]\Delta X(t_i^{(k)}).$$

As in the proof of Theorem 1 we have

$$\begin{aligned}(S^r_{\theta''}-S^r_{\theta'})\Omega &= \sum_{k=1}^{m}\sum_{i=1}^{l_k}\big[f(t^{(k)}_{i-1})-f(t_{k-1})\big]\Delta X(t^{(k)}_i)\Omega \\ &= \sum_{k=1}^{m}\sum_{i=1}^{l_k}\big[f(t^{(k)}_{i-1})-f(t_{k-1})\big](P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}})X(t)\Omega \\ &= (\sigma^r_{\theta''}-\sigma^r_{\theta'})X(t)\Omega,\end{aligned}$$

and thus

$$\|(S^r_{\theta''}-S^r_{\theta'})\Omega\|_2^2 = \sum_{k=1}^{m}\sum_{i=1}^{l_k}\big\|(P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}})\big[f(t^{(k)}_{i-1})-f(t_{k-1})\big]X(t)\Omega\big\|^2_{\mathcal{H}}$$

by the orthogonality of $P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}}$ and $P_{t^{(k)}_j}-P_{t^{(k)}_{j-1}}$ for $i\neq j$. Furthermore

$$\begin{aligned}&\big\|(P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}})\big[f(t^{(k)}_{i-1})-f(t_{k-1})\big]X(t)\Omega\big\|^2_{\mathcal{H}} \\ &\quad=\Big\langle (P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}})\big|f(t^{(k)}_{i-1})-f(t_{k-1})\big|^2\big(P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}}\big)X(t)\Omega, X(t)\Omega\Big\rangle,\end{aligned}$$

and since $|t^{(k)}_{i-1}-t_{k-1}|<\delta$, we have

$$|f(t^{(k)}_{i-1})-f(t_{k-1})|^2\leqslant \frac{\varepsilon^2}{4\|X(t)\Omega\|^2_{\mathcal{H}}}\mathbf{1},$$

giving

$$\big(P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}}\big)\big|f(t^{(k)}_{i-1})-f(t_{k-1})\big|^2\big(P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}}\big)\leqslant \frac{\varepsilon^2}{4\|X(t)\Omega\|^2_{\mathcal{H}}}\big(P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}}\big).$$

This yields the estimate

$$\begin{aligned}&\big\|\big(P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}}\big)\big[f\big(t^{(k)}_{i-1}\big)-f(t_{k-1})\big]X(t)\Omega\big\|^2_{\mathcal{H}} \\ &\quad=\big\langle\big(P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}}\big)\big|f(t^{(k)}_{i-1})-f(t_{k-1})\big|^2\big(P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}}\big)X(t)\Omega, X(t)\Omega\big\rangle \\ &\quad\leqslant \frac{\varepsilon^2}{4\|X(t)\Omega\|^2_{\mathcal{H}}}\big\|\big(P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}}\big)X(t)\Omega\big\|^2_{\mathcal{H}}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\big\|\big(S^r_{\theta''}-S^r_{\theta'}\big)\Omega\big\|^2_{\mathcal{H}} &= \sum_{k=1}^{m}\sum_{i=1}^{l_k}\big\|\big(P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}}\big)\big[f(t^{(k)}_{i-1})-f(t_{k-1})\big]X(t)\Omega\big\|^2_{\mathcal{H}} \\ &\leqslant \frac{\varepsilon^2}{4\|X(t)\Omega\|^2_{\mathcal{H}}}\sum_{k=1}^{m}\sum_{i=1}^{l_k}\big\|\big(P_{t^{(k)}_i}-P_{t^{(k)}_{i-1}}\big)X(t)\Omega\big\|^2_{\mathcal{H}}\end{aligned}$$

$$= \frac{\varepsilon^2}{4\|X(t)\Omega\|^2_{\mathcal{H}}} \left\| \sum_{k=1}^{m} \sum_{i=1}^{l_k} \big(P_{t_i^{(k)}} - P_{t_{i-1}^{(k)}}\big) X(t)\Omega \right\|^2_{\mathcal{H}}$$

$$= \frac{\varepsilon^2}{4\|X(t)\Omega\|^2_{\mathcal{H}}} \big\|(P_t - P_0)X(t)\Omega\big\|^2_{\mathcal{H}}$$

$$\leqslant \frac{\varepsilon^2}{4\|X(t)\Omega\|^2_{\mathcal{H}}} \big\|X(t)\Omega\big\|^2_{\mathcal{H}} = \frac{\varepsilon^2}{4}. \tag{3}$$

Let now θ_1 and θ_2 be arbitrary partitions of $[0,t]$ such that $\|\theta_1\| < \delta$, $\|\theta_2\| < \delta$, and let $\theta'' = \theta_1 \cup \theta_2$. Then we have by (3)

$$\big\|\big(S^r_{\theta''} - S^r_{\theta_1}\big)\Omega\big\|_{\mathcal{H}} < \frac{\varepsilon}{2} \quad \text{and} \quad \big\|\big(S^r_{\theta''} - S^r_{\theta_2}\big)\Omega\big\|_{\mathcal{H}} < \frac{\varepsilon}{2},$$

so

$$\big\|\big(S^r_{\theta_1} - S^r_{\theta_2}\big)\Omega\big\|_{\mathcal{H}} < \varepsilon,$$

showing that the net $\{S^r_\theta \Omega\}$ is Cauchy. Thus there exists $\lim_{\|\theta\|\to 0} S^r_\theta \Omega$ and the rest of the proof is the same as that of Theorem 1. □

3 Integrals as Operators on $\mathcal{H}$—the Tracial Case

Let us now assume that ω is a normal tracial state. Recall that the Lebesgue space $L^2(\mathcal{A},\omega)$ is formally defined as the completion of $\mathcal{A}$ with respect to the norm

$$\|x\|_2 = \big[\omega(x^*x)\big]^{1/2} = \|x\Omega\|_{\mathcal{H}},$$

and may be realized as a space of densely defined closed operators affiliated with $\mathcal{A}$ such that Ω belongs to their domains.

For the von Neumann subalgebras $\mathcal{A}_t$ the normal ω-invariant conditional expectations $\mathbb{E}_t\colon \mathcal{A} \to \mathcal{A}_t$ extend to orthogonal projections (denoted by the same letter) from $L^2(\mathcal{A},\omega)$ onto $L^2(\mathcal{A}_t,\omega)$. If we define an operator P_t on $\mathcal{H}$ by

$$P_t(X\Omega) = (\mathbb{E}_t X)\Omega, \quad X \in L^2(\mathcal{A},\omega), \tag{4}$$

then P_t is an orthogonal projection from $\mathcal{A}'_t$. (The definition above is the same as that given by (1) for $X \in \mathcal{A}$.) For any $x \in \mathcal{A}$, $A \in L^2(\mathcal{A},\omega)$ the operators xA and Ax belong to $L^2(\mathcal{A},\omega)$, consequently we may again consider the integral sums

$$\sum_{k=1}^{m} f(t_{k-1})\Delta X(t_k), \qquad \sum_{k=1}^{m} \Delta X(t_k) f(t_{k-1}),$$

where $f\colon [0,\infty) \to \mathcal{A}$, $X\colon [0,\infty) \to L^2(\mathcal{A},\omega)$, as operators on $\mathcal{H}$. We shall use the notation

$$S^r_\theta(f,X) = \sum_{k=1}^{m} f(t_{k-1})\Delta X(t_k), \qquad S^l_\theta(f,X) = \sum_{k=1}^{m} \Delta X(t_k) f(t_{k-1}),$$

for the integral sums, to indicate their dependence on f and X. Then since $S^r_\theta(f,X)$ and $S^l_\theta(f,X)$ are affiliated with $\mathcal{A}$, we have an explicit description of their actions on $\mathcal{A}'\Omega$ as

$$S^r_\theta(f,X)(x'\Omega) = x' S^r_\theta(f,X)\Omega, \qquad S^l_\theta(f,X)(x'\Omega) = x' S^l_\theta(f,X)\Omega.$$

Theorem 3 *Let $(X(t) : t \geqslant 0)$ be a martingale in $L^2(\mathcal{A}, \omega)$ and let $f : [0, \infty) \to \mathcal{A}$ be either monotone or norm continuous. Then for each $t > 0$ there exist integrals $\int_0^t f\, dX$ and $\int_0^t dX\, f$ as elements of $L^2(\mathcal{A}, \omega)$. Moreover, the $L^2(\mathcal{A}, \omega)$-processes $(Y(t) : t \geqslant 0)$, $(Z(t) : t \geqslant 0)$ defined by*

$$Y(t) = \int_0^t dX\, f, \qquad Z(t) = \int_0^t f\, dX$$

are martingales.

Proof The existence of the integral $\int_0^t f\, dX$ is proved exactly as in Theorems 1 and 2 upon observing that according to formula (4) we have

$$\big[X(t_k) - X(t_{k-1})\big]\Omega = \big[\mathbb{E}_{t_k} X(t) - \mathbb{E}_{t_{k-1}} X(t)\big]\Omega = (P_{t_k} - P_{t_{k-1}})X(t)\Omega.$$

It follows that the net $\{S_\theta^r(f, X)\Omega\}$ is Cauchy, and since

$$\big\| S_\theta^r(f, X)\big\|_{\mathcal{H}} = \big\| S_\theta^r(f, X)\big\|_2,$$

$\{S_\theta^r(f, X)\}$ converges in $\|\cdot\|_2$-norm, and thus its limit is an element of $L^2(\mathcal{A}, \omega)$.

For $S_\theta^l(f, X)$ we have

$$S_\theta^l(f, X) = \big[S_\theta^r(f^*, X^*)\big]^*;$$

so we obtain

$$\big\| S_\theta^l(f, X)\big\|_2 = \big\|\big[S_\theta^r(f^*, X^*)\big]^*\big\|_2 = \big\| S_\theta^r(f^*, X^*)\big\|_2,$$

because Ω is tracial. Since f^* satisfies the same assumptions as f, and $(X(t)^* : t \geqslant 0)$ is also an $L^2(\mathcal{A}, \omega)$-martingale, we obtain the convergence of $\{S_\theta^l(f, X)\}$ in $\|\cdot\|_2$-norm.

Now we shall show that $(Y(t) : t \geqslant 0)$ is a martingale. Fix $t > 0$ and take an arbitrary $s < t$. We have

$$\int_0^t dX\, f = \lim_{\|\theta\| \to 0} S_\theta^l.$$

We may assume that s is one of the points of each partition $\theta = \{0 = t_0 < t_1 < \cdots < t_m = t\}$, say $s = t_k$. Then we have

$$\begin{aligned} \mathbb{E}_s S_\theta^l &= \mathbb{E}_s \left(\sum_{i=1}^k \big[X(t_i) - X(t_{i-1})\big] f(t_{i-1}) + \sum_{i=k+1}^m \big[X(t_i) - X(t_{i-1})\big] f(t_{i-1}) \right) \\ &= \sum_{i=1}^k \mathbb{E}_s \big[X(t_i) - X(t_{i-1})\big] f(t_{i-1}) + \sum_{i=k+1}^m \mathbb{E}_s \big[X(t_i) - X(t_{i-1})\big] f(t_{i-1}). \end{aligned}$$

For $i \leqslant k$ we have $t_i \leqslant s$, and thus

$$\mathbb{E}_s[X(t_i) - X(t_{i-1})] f(t_{i-1}) = [X(t_i) - X(t_{i-1})] f(t_{i-1}),$$

while for $i > k$ we have $t_{i-1} \geqslant s$, and thus

$$\begin{aligned} \mathbb{E}_s \big[X(t_i) - X(t_{i-1})\big] f(t_{i-1}) &= \mathbb{E}_s \mathbb{E}_{t_{i-1}} \big[X(t_i) - X(t_{i-1})\big] f(t_{i-1}) \\ &= \mathbb{E}_s \big(\mathbb{E}_{t_{i-1}} \big[X(t_i) - X(t_{i-1})\big]\big) f(t_{i-1}) = 0 \end{aligned}$$

by the martingale property. Consequently,

$$\mathbb{E}_s S_\theta^l = \sum_{i=1}^{k} \big[X(t_i) - X(t_{i-1})\big] f(t_{i-1}). \tag{5}$$

But the sum on the right hand side of (5) is an integral sum for the integral $\int_0^s dX\, f$, and passing to the limit in (5) yields

$$\mathbb{E}_s \int_0^t dX\, f = \int_0^s dX\, f,$$

which shows that $(Y(t))$ is a martingale. Analogously for $(Z(t) : t \geqslant 0)$. □

Remarks 1. The formulas

$$\left(\int_0^t f\, dX\right)(x'\Omega) = x' \lim_\theta S_\theta^r \Omega, \qquad \left(\int_0^t dX\, f\right)(x'\Omega) = x' \lim_\theta S_\theta^l \Omega$$

describe explicitly the actions of the integrals on $\mathcal{A}'\Omega$.

2. Let us notice that an attempt to define Lebesgue type integrals above, e.g., along the lines of [4] or [10], would be successful only in the simple case of norm-continuous f, and even then under the additional assumption of left- or right-continuity in $\|\cdot\|_2$-norm of the martingale $(X(t))$. The reason for this is that this type of integral is defined for μ-measurable functions $f \in L^2([0,\infty), \mu, \mathcal{A})$, where μ is a measure defined by

$$\mu([a,b)) \text{ or } \mu((a,b]) = \omega(|X(b) - X(a)|^2) = \omega(|X(b)|^2) - \omega(|X(a)|^2).$$

The continuity of the martingale allows then the extension of μ from the intervals to the Borel sets. For the case of monotone f the failure of the definition of the integral as Lebesgue type is most strikingly seen when the function f is increasing projection-valued. To be more concrete, assume that e is a spectral measure with support $[0,\infty)$, and put $f(t) = e([0,t])$. Then for $s < t$, $f(t) - f(s)$ is a non-zero projection, so $\|f(t) - f(s)\| = 1$, which means that f is not norm continuous either from the left or from the right at any point, while a μ-measurable function must be norm-continuous on some compact set.

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